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DIVERGENCE AND FLUTTER OF A CANTILEVER ROD WITH AN INTERMEDIATE SPRING SUPPORT

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Abstract—The equation of motion in matrix form of a cantilever Euler beam subject to a tip-concentrated follower force at the free end is formulated based on the Lagrangian approach and the assumed mode method. The non-conservative nature of the system is identified by the non-symmetric matrix in the equation of motion. The beam is assumed to rest on an intermediate spring support. Jump phenomenon for the critical follower load is found to occur for both variations in the support location as well as the stiffness of the spring support. Detailed history of the changes in the load–frequency diagram due to variation in the support location and the stiffness of the spring support is presented to explain the occurrence of the jump phenomenon. An interesting finding is that the rod is found to be extremely unstable for a certain combination of spring stiffness and support location for a cantilever rod.

INTRODUCTION

The stability of non-conservative systems has been studied extensively since the early works by Bolotin (1963, 1964) and Ziegler (1968). A well-known problem for the stability of non-conservative systems is the dynamic behavior of a rod subject to tip-concentrated or distributed follower forces. Follower forces are forces whose lines of action are affected by the deformation of the rod. These forces are non-conservative as the resulting virtual work cannot be represented by a potential function. A comprehensive discussion of this subject with a related list of references can be found in the book by Leipholz (1980).

Exact solutions may be difficult or not possible at all for most of these problems. Stability analyses are often based on approximation techniques such as Galerkin's method (for example, Leipholz, 1980), the Ritz stationary functional method (Levinson, 1966), the finite difference method, the finite element method and many other discretization methods (for example, De Rosa and Franciosi, 1990; Lee and Kuo, 1990). Galerkin's method has been shown by Leipholz (1980) to be in general convergent. However, the convergence of the load–frequency curves of the corresponding non-conservative problem is not always uniform. A two-term Galerkin's approximation for a clamped–clamped beam under uniformly distributed tangential follower loads was found to give erroneous results. A four-term approximation, however, did lead to correct load–frequency curves. The finite difference method has been used by Leipholz (1980) for analyzing a variety of non-conservative problems. A feature of this method is that many pivot points are required to obtain accurate load–frequency curves. Guran and Rimrott (1989) presented a multi-local difference method based on the Hermitian method for enhancing the accuracy and the rate of convergence of the finite difference method. For a classical Beck's rod with uniform cross-section, a discretization of 50 elements was found to be required for the convergence of the first critical flutter load using this improved finite difference method. A different form of the discretization method was presented by De Rosa and Franciosi (1990) for analyzing the effect of an intermediate support on the dynamic behavior of cantilever beams subjected to follower forces. A variety of beam configurations such as the generalized Beck's rod, the generalized Leipholz's rod, the Pflüger system and the generalized Hauger's rod was analyzed in their work. In all of these non-conservative systems, transition from flutter to divergence was found to occur for a specific location of the intermediate support depending on the system. Moreover, the transition is in the form of an abrupt variation, or discontinuity in the critical load, a feature known as the "jump phenomenon". The same phenomenon was reported earlier by Zorii and Chernukha (1971) for the same problem and in a series of papers by Kounadis (1980, 1981, 1983) and Kounadis and Economou (1980) for other related problems. Elishakoff and Hollkamp (1987) studied the same problem of Zorii and Chernukha (1971) using computerized symbolic algebra in conjunction with the two-term Galerkin's method. The jumped discontinuity for the transition from flutter to divergence with respect to changes in the location of the intermediate support was also reported. However, in subsequent work by Elishakoff and Lottati (1988), an exact solution was attempted by citing the findings of Hauger and Leonard (1978) that approximation might yield results which were both qualitatively and quantitatively different from the exact solutions. The findings for a simple support-free rod and a clamp-free rod on an intermediate support showed that there was no jump phenomenon, contrary to the findings of all the related earlier reported works and the subsequent study by De Rosa and Franciosi (1990) using the discretization method. In view of these reported studies, a pertinent question is: is there a jumped discontinuity in the critical loads from flutter to divergence when the intermediate support is shifted slightly at the critical location corresponding to the transition?

In the present paper, the equation of motion in matrix form of a cantilever Euler beam resting on an intermediate spring support subject to a tip-concentrated follower force at the free end is formulated based on the Lagrangian approach and the assumed mode method. The present approach, similar to other discretization methods, results in an equation of motion for a rod subject to a follower force to be different from the equation of motion of a rod subject to a conservative axial load. One can identify the non-conservative nature of the system easily by the presence of a non-symmetric matrix in the equation of motion. Convergence of the lowest critical flutter load is first checked for a cantilever Beck's rod without the spring support. The effects of an intermediate spring support on the load–frequency curves in terms of its location and the spring stiffness are then examined for a cantilever beam. It should be pointed out that a related problem for the effect of a spring support on the stability of pipes conveying fluid was presented by Sugiyama *et al.* (1985) based on the Galerkin's method. A comprehensive list of references for the stability of pipes conveying fluid can be found in the recent review article by Païdoussis and Li (1993).

THEORY AND FORMULATIONS

Figure 1 shows a cantilever rod of length l of uniform cross-section and mass m per unit length. The rod is resting on an intermediate spring support located at a distance s from the clamped end of the rod. The spring support is assumed to be linear with spring stiffness k. The rod is subjected to a follower force P at the free end of the rod. The deflection of the rod is assumed to be small for the behavior to be governed by Euler's beam theory. A set of right-handed mutually perpendicular unit vectors, **i** and **j**, is assumed to be fixed in the undeformed rod with the **i** vector parallel to the undeformed neutral axis of the rod. The deflection for a point x on the rod is denoted by $w(x, t)$ with t denoting the time.

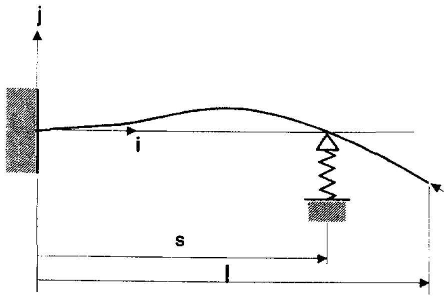

Fig. 1. A cantilever beam on an intermediate spring support subject to a follower force.

In some of the reported works based on the finite difference method or Galerkin's method for analyzing the stability of a cantilever rod subject to a tip-concentrated follower load, the starting point of the formulation is always the governing differential equation (For example, Elishakoff and Lottati, 1988, Guran and Rimrott, 1989):

$$EI\frac{\partial^4 w}{\partial x^4}+P\frac{\partial^2 w}{\partial x^2}+m\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

with the associated boundary conditions

$$w=\frac{\partial w}{\partial x}=0 \quad \text{at} \quad x=0 \tag{2}$$

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^3 w}{\partial x^3}=0 \quad \text{at} \quad x=l. \tag{3}$$

An interesting observation is that the same equation of motion and the associated boundary conditions can also be obtained for a cantilever rod subject to a conservative axial load P (see, for example, Rao, 1990). Therefore, the above equation of motion together with the associated boundary conditions does not describe a non-conservative system as both the equation of motion and the boundary conditions appear to be self-adjoint. The second boundary condition related to the transverse shear force at the free end is indeed incorrect and should be given by (Kounadis, 1983)

$$EI\frac{\partial^3 w}{\partial x^3}+P\frac{\partial w}{\partial x}=0 \quad \text{at} \quad x=l. \tag{4}$$

It is the non self-adjoint nature of the above boundary condition that causes the system to be non-conservative.

In the following derivation using the Lagrangian approach, the equation of motion in matrix form for a rod subject to a follower load will be shown to be different from the equation of motion of a rod subject to a conservative axial load, just like the equation of motion derived by other discretization methods or the finite element method (for example, De Rosa and Franciosi, 1990). One can identify the non-conservative nature of the system easily by the presence of a non-symmetric matrix in the equation of motion.

The elastic strain energy of the rod due to bending is

$$V_\varepsilon=\tfrac{1}{2}EI\int_0^l\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x \tag{5}$$

where E and I are the Young's modulus and the central principal second moment of area of the cross section of the rod.

The total kinetic energy T of the rod is

$$T=\tfrac{1}{2}m\int_0^L \dot{w}^2\,\mathrm{d}x \tag{6}$$

where the variable $\dot{w}$ is defined by $\mathrm{d}w/\mathrm{d}t$.

The potential energy of the linear spring support located at $x=s$ is given by

$$V_s = \tfrac{1}{2}kw^2(x = s, t). \tag{7}$$

The work done by the follower force can be divided into two parts. The axial component of the follower force in the **i** direction, parallel to the neutral axis of the undeformed rod, is equal to P for small deflection, neglecting quantities of the order of (w^2) and smaller. The transverse component in the **j** direction, normal to the neutral axis of the undeformed rod, is given by $P(\partial w/\partial x)(x = l, t)$. Therefore, the potential energy due to the axial component of the follower force is

$$V_a = -\frac{1}{2}\int_0^l P\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x. \tag{8}$$

On the other hand, the transverse component of the follower force is a non-conservative force and no potential energy can be defined for this force component. The virtual work of this transverse component of the follower force is

$$\delta W = -P\frac{\partial w}{\partial x}(x = l, t)\delta w(x = l, t). \tag{9}$$

Using the assumed mode method, the quantity w can be expressed as

$$w(x, t) = \sum_{i=1}^{n} q_i(t)\phi_i(x) \tag{10}$$

where ϕ_i are spatial functions that satisfy the prescribed geometric boundary conditions for the two ends of the beam. It should be noted that, just like the Rayleigh–Ritz method, the functions are not required to satisfy the natural boundary conditions related to the bending moment and transverse shear force at the free end of the rod. For the present problem, the normalized beam functions for a clamp-free beam are used for the numerical simulations.

The assumed form of w enables the kinetic energy, T, the strain energy, V_ε, the potential energy of the spring support, V_k, the potential energy of the axial component of the follower force, V_a, and the virtual work due to the transverse component of the follower force, δW, to be expressed in matrix form as follows:

$$T = \tfrac{1}{2}m\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} \tag{11}$$

$$V_\varepsilon = \tfrac{1}{2}EI\mathbf{q}^T\mathbf{K}\mathbf{q} \tag{12}$$

$$V_k = \tfrac{1}{2}k\mathbf{q}^T\mathbf{S}\mathbf{q} \tag{13}$$

$$V_a = -\tfrac{1}{2}P\mathbf{q}^T\mathbf{A}\mathbf{q} \tag{14}$$

$$\delta W = -P\delta\mathbf{q}^T\mathbf{D}\mathbf{q}. \tag{15}$$

The matrices **M, K, S, A**, and **D** are matrices defined as

$$(\mathbf{M})_{ij} = \int_0^L \phi_i\phi_j\,\mathrm{d}x \tag{16}$$

$$(\mathbf{K})_{ij} = \int_0^L \phi_i''\phi_j''\,\mathrm{d}x \tag{17}$$

$$(\mathbf{S})_{ij} = \phi_i(x = s)\phi_j(x = s) \tag{18}$$

$$(\mathbf{A})_{ij} = \int_0^L \phi_i'\phi_j' \, \mathrm{d}x \tag{19}$$

$$(\mathbf{D})_{ij} = \phi_i(x = l)\phi_j'(x = l). \tag{20}$$

The functions ϕ_i' and ϕ_i'' denote the first and second derivatives of ϕ_i with respect to x. The vectors $\mathbf{q}$ and $\dot{\mathbf{q}}$ are $n \times 1$ column vectors. All the matrices except $\mathbf{D}$ are symmetric matrices. The resulting linearized Euler–Lagrange equation in matrix form is given by

$$m\mathbf{M}\ddot{\mathbf{q}} + [EI\mathbf{K} + k\mathbf{S} - P(\mathbf{A} - \mathbf{D})]\mathbf{q} = 0. \tag{21}$$

For a rod subject to a conservative axial force P, the term involving the non-symmetric matrix $\mathbf{D}$ will not appear in the above equation. The non-symmetric matrix $\mathbf{D}$ causes the equation of motion to be different from that of a rod subject to conservative forces. This matrix equation involving an unsymmetric matrix is similar in form to the matrix equation presented by De Rosa and Franciosi (1990) using a discretization technique, and Leung (1988) for a Timoshenko beam subject to follower forces using the dynamic stiffness method.

For simplicity, the following dimensionless quantities are introduced

$$\tau = t\sqrt{\frac{EI}{ml^4}}, \quad \xi = \frac{x}{l}, \quad \bar{P} = \frac{Pl^2}{EI}, \quad \bar{k} = \frac{kl^3}{EI}, \quad \bar{s} = \frac{s}{l}. \tag{22}$$

The dimensionless w is given by

$$\bar{w} = \frac{w}{l} = \sum_{i=1}^{n} \bar{q}_i(\tau)\phi_i(\xi). \tag{23}$$

The resulting dimensionless equation of motion is

$$\bar{\mathbf{M}}\ddot{\bar{\mathbf{q}}} + (\bar{\mathbf{K}} + \bar{k}\bar{\mathbf{S}} - \bar{P}(\bar{\mathbf{A}} - \bar{\mathbf{D}})\bar{\mathbf{q}} = 0. \tag{24}$$

The matrices in the above equation are defined as

$$(\bar{\mathbf{M}})_{ij} = \int_0^1 \phi_i\phi_j \, \mathrm{d}\xi \tag{25}$$

$$(\bar{\mathbf{K}})_{ij} = \int_0^1 \phi_i''\phi_j'' \, \mathrm{d}\xi \tag{26}$$

$$(\bar{\mathbf{S}})_{ij} = \phi_i(\xi = \bar{s})\phi_j(\xi = \bar{s}) \tag{27}$$

$$(\bar{\mathbf{A}})_{ij} = \int_0^1 \phi_i'\phi_j' \, \mathrm{d}x \tag{28}$$

$$(\bar{\mathbf{D}})_{ij} = \phi_i(\xi = 1)\phi_j'(\xi = 1). \tag{29}$$

The matrix $\bar{\mathbf{M}}$ is an identity matrix due to the orthogonality of the normalized beam functions. The corresponding eigenvalue problem of the above equation is given by

$$\det|\bar{\mathbf{K}}+\bar{k}\bar{\mathbf{S}}-\bar{P}(\bar{\mathbf{A}}-\bar{\mathbf{D}})-\omega^2\mathbf{M}| = 0 \tag{30}$$

where ω are the dimensionless natural frequencies of the dynamical system. These dimensionless natural frequencies can be easily computed using any numerical package for matrix computations.

The normalized beam functions for a beam clamped at $\xi = 0$ and free at $\xi = 1$ are

$$\phi_i(\xi) = \sin\beta\xi - \sinh\beta\xi - \gamma(\cos\beta\xi - \cosh\beta\xi) \tag{31}$$

where β are the eigenvalues that satisfy the characteristic equation for the beam vibration with one end clamped and the other end free and

$$\gamma = \frac{\sin\beta+\sinh\beta}{\cos\beta+\cosh\beta}. \tag{32}$$

NUMERICAL RESULTS AND DISCUSSION

The convergence of the present formulation is first examined for a Beck's rod (Beck, 1952) without the intermediate spring support. The load–frequency curves computed using 2, 3, 10, and 20-term approximations for $\bar{w}$ (n = 2, 3, 10 and 20) are shown in Fig. 2. It can be seen that the load–frequency curve is almost converged for $n = 3$. However, the first critical flutter load, corresponding to the apex of the dome-shaped curve where coalescence of the eigenvalues occurs, is found to be 17.60, a value smaller than the reported results based on the finite difference method (Guran and Rimrott, 1989) and the exact value (20.05) reported by Beck (1952). At the point of coalescence, the two positive real eigenvalues coalesce into a pair of complex eigenvalues with a common real part. The real part of this pair of complex eigenvalues is shown as a solid line separating from the lower dome-shaped curve at the point of coalescence. The differences between the present critical flutter load and the reported value could be due to the differences and the possible errors in the handling of the boundary conditions and the formulation of the equation of motion discussed in the previous section. For the present formulation using the assumed mode method, the natural boundary conditions at the free end of the rod need not be satisfied. The effect of the follower force is contained in the potential energy of the axial component

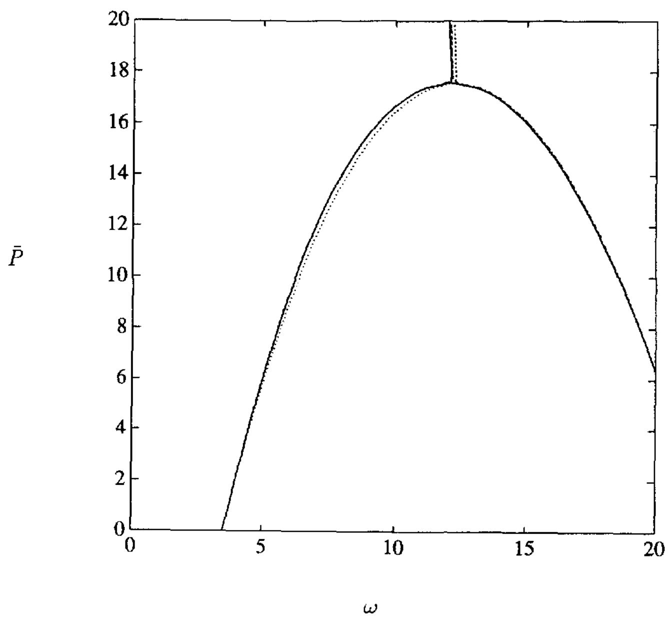

Fig. 2. The load–frequency curves for a Beck's rod without any intermediate spring support. (——) $n = 20$, (- - - -) $n = 10$, (- · - · - ·) $n = 3$, (· · · · · ·) $n = 2$.

of the follower force and the virtual work of the transverse portion of the follower force. It is the virtual work of the transverse portion of the follower force that gives rise to the non-symmetric matrix in the equation of motion. For the reported finite difference method, the zero transverse shear force condition enforced at the free end of the rod does not satisfy the boundary condition due to the presence of the follower force.

The effects of intermediate spring supports are examined for two cases with $\bar{s} = 0.4$ and 0.55. For each case, the stiffness of the spring is varied between 1 and 10^5. The load–frequency diagrams for these two locations of spring support are shown respectively in Figs 3 and 4. It can be seen from these diagrams that for the stiffness of the spring $\bar{k}$ equal to or larger than 10^4, the load–frequency curves remain almost unchanged. The rod can be viewed as a cantilever rod with an intermediate rigid support as the stiffness of the spring is large. For the load–frequency diagrams shown in Fig. 3 with $\bar{s} = 0.4$, the lowest critical loads are the critical loads for flutter except for the case of $\bar{k} = 10^2$. For this special case of $\bar{k} = 10^2$, the first frequency curve intersects the vertical axis ($\omega = 0$) at about $\bar{P} = 40$, corresponding to the lowest critical load for divergence. Therefore, for $\bar{k}$ varied between 10^1 and 10^3, the mode of instability changes from flutter to divergence and then back to flutter. For a rod

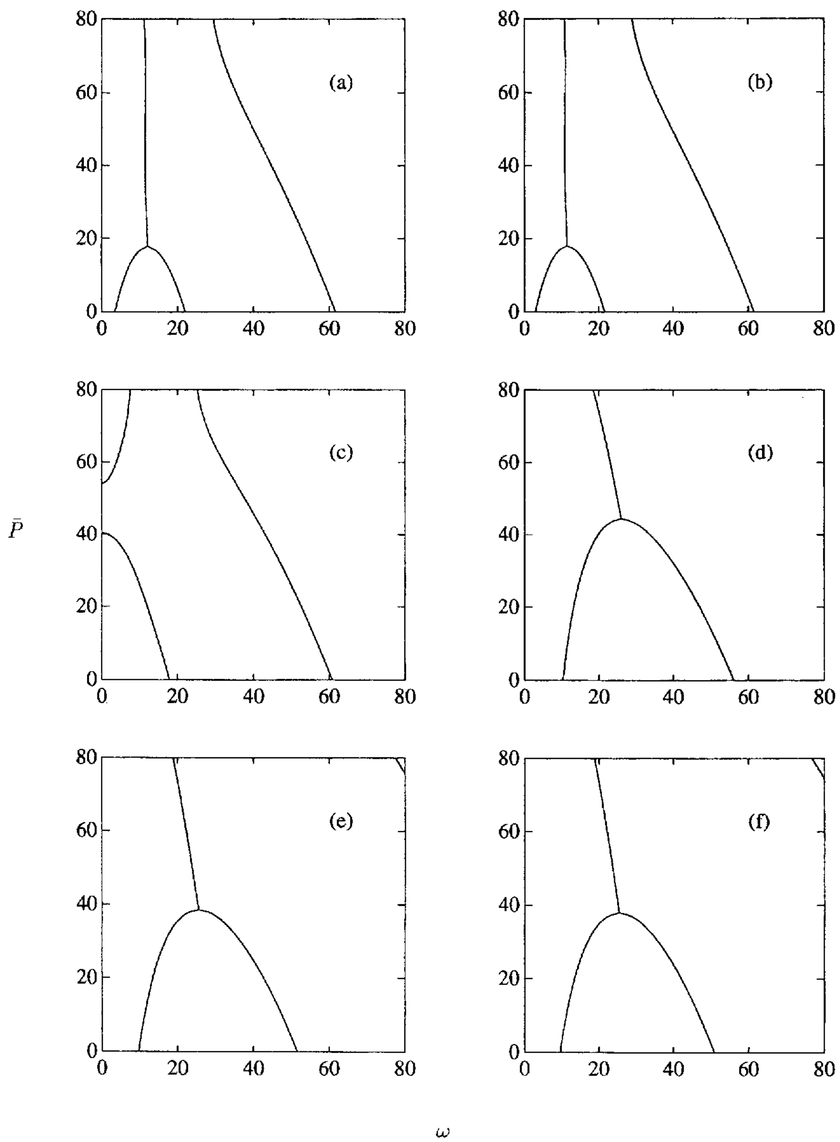

Fig. 3. The load-frequency curves for a cantilever rod with an intermediate spring support at $\bar{s} = 0.4$. (a) $\bar{k} = 10^0$; (b) $\bar{k} = 10^1$; (c) $\bar{k} = 10^2$; (d) $\bar{k} = 10^3$; (e) $\bar{k} = 10^4$; (f) $\bar{k} = 10^5$.

on an intermediate spring support with large stiffness, the mode of instability is by flutter, consistent with the reported results by Elishakoff and Lottati (1988) although the critical flutter load is slightly smaller than the corresponding reported results based on exact formulation, probably due to the same reasons discussed in the previous section.

For the load–frequency diagrams shown in Fig. 4 with $\bar{s} = 0.55$, the mode of instability is flutter for the intermediate spring support with small stiffness. The mode of instability changes from flutter to divergence for $\bar{k}$ varied between 10^1 and 10^2. For $\bar{k}$ equal to and larger than 10^4, the mode of instability is divergence, consistent with the reported results (Elishakoff and Lottati, 1988) for a rod on an intermediate rigid support. The first critical divergence load is again slightly smaller than the corresponding value reported by Elishakoff and Lottati (1988). Comparing Figs 3 and 4, it can be seen that the mode of instability changes from flutter to divergence when the location of the intermediate spring support of large stiffness shifted from $\bar{s} = 0.4$ to $\bar{s} = 0.55$. The history of this transition is shown in Fig. 5 for $\bar{k} = 10^5$. When $\bar{s} = 0.5$, the mode of instability is flutter as indicated by the dome-shaped curves in part (a) of Fig. 5. The first flutter load appears to be smaller than the first divergence load in that diagram. When the intermediate support is shifted slightly to the

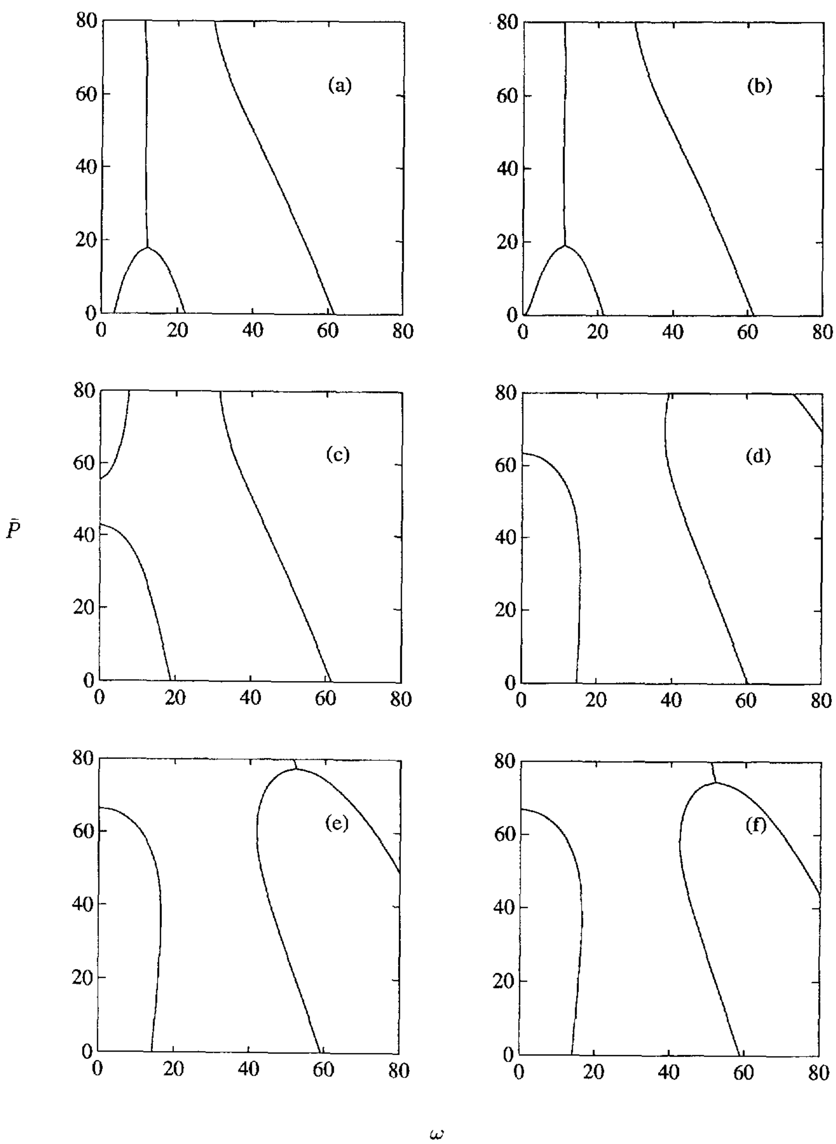

Fig. 4. The load–frequency curves for a cantilever rod with an intermediate spring support at $\bar{s} = 0.55$. (a) $\bar{k} = 10^0$; (b) $\bar{k} = 10^1$; (c) $\bar{k} = 10^2$; (d) $\bar{k} = 10^3$; (e) $\bar{k} = 10^4$; (f) $\bar{k} = 10^5$.

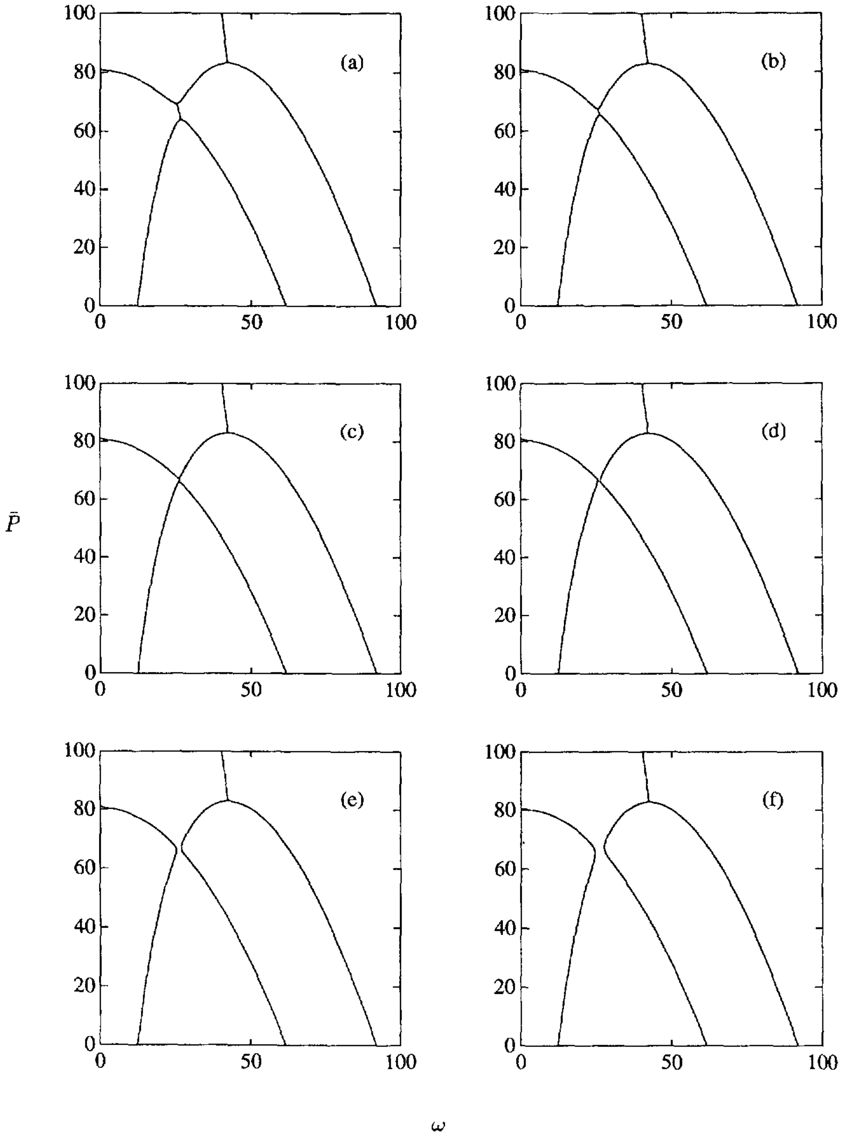

Fig. 5. The load–frequency curves for a cantilever rod with an intermediate spring support with $\bar{k} = 10^5$. (a) $\bar{s} = 0.5$; (b) $\bar{s} = 0.5005$; (c) $\bar{s} = 0.50055$; (d) $\bar{s} = 0.5006$; (e) $\bar{s} = 0.5007$; (f) $\bar{s} = 0.501$.

right, the two parts of the curves shown in Figs 5(a) and 5(b) begin to move away from each other. When $\bar{s}$ is slightly larger than 0.50055 shown in Fig. 5(d), the dome-shaped curve disappears and the first divergence load becomes the first critical load. This critical support position of $\bar{s} \approx 0.5$ is consistent with the reported value by Elishakoff and Lottati (1988). It can be seen from these diagrams in Fig. 5 that there is no jump in the first divergence load when $\bar{s}$ is varied slightly about 0.5. However, the first divergence load is very much larger than the first flutter load before the disappearance of the dome-shaped curve. The disappearance of the dome-shaped curve therefore causes the jump phenomenon in the first critical load for the transition of the mode of instability from flutter to divergence. The same phenomenon is also reported by De Rosa and Franciosi (1990).

The history of the variation of the mode of instability from flutter to divergence for $\bar{s} = 0.4$ shown earlier in Fig. 3 is examined in Fig. 6. When the stiffness of the spring $\bar{k}$ is increased slightly from 20, the dome-shaped curve begins to move slightly to the left. At $\bar{k} = 28$, the lower left hand corner of the curve touches the vertical axis ($\omega = 0$), causing the mode of instability to change from flutter to divergence. An interesting observation is that the first divergence load is almost zero for this combination of support location and

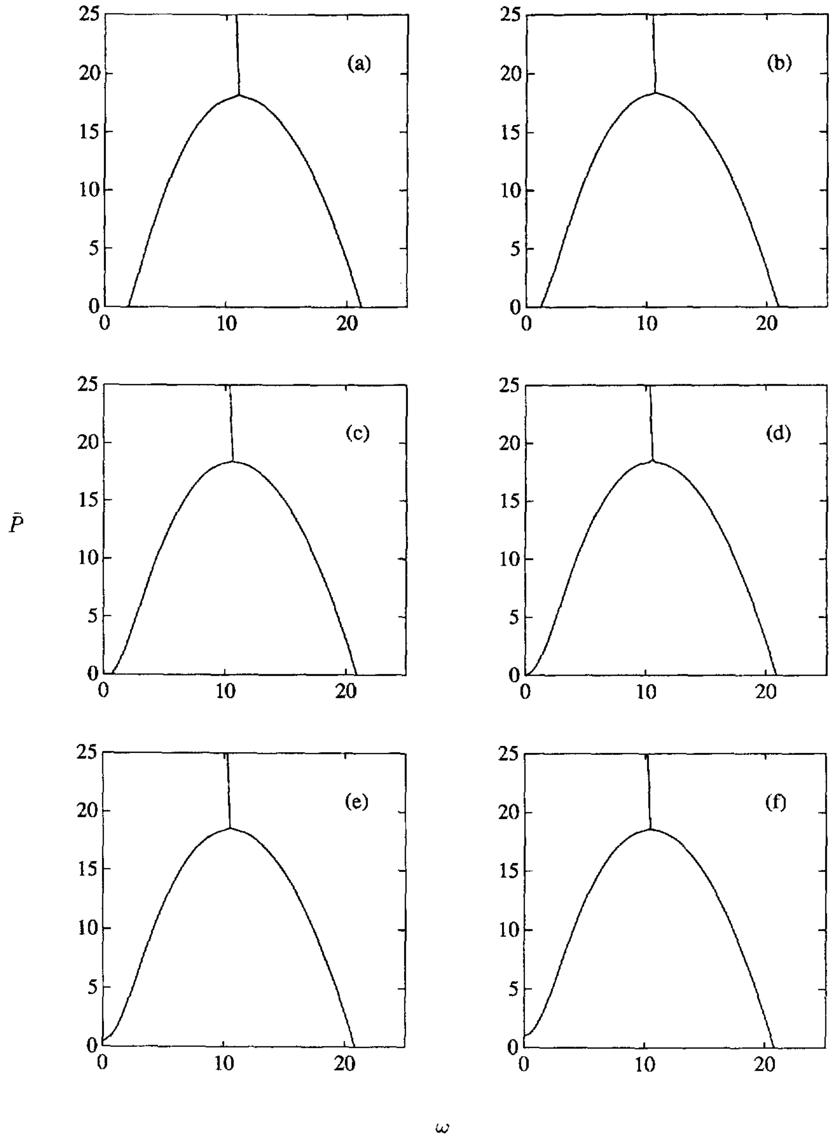

Fig. 6. The load–frequency curves for a cantilever rod with an intermediate spring support at $\bar{s} = 0.4$. (a) $\bar{k} = 20$; (b) $\bar{k} = 25$; (c) $\bar{k} = 27$; (d) $\bar{k} = 28$; (e) $\bar{k} = 29$; (f) $\bar{k} = 30$.

spring stiffness. Therefore, the rod is extremely unstable for such a combination. This type of load–frequency curve is not reported in the book by Leipholz (1980). When the stiffness of the spring is further increased, the first critical load for divergence is found to increase steadily. The transition of the mode of instability from divergence to flutter for $\bar{k}$ between 10^2 and 10^3 for $\bar{s} = 0.4$ shown in Fig. 3 is examined in Fig. 7. Once again, there are no drastic changes in the first divergence load when $\bar{k}$ is increased slightly from $\bar{k} = 200$. The jump phenomenon for the first critical load is due to the re-appearance of the dome-shaped curve when $\bar{k}$ is slightly larger than 219. As the first flutter load due to the presence of the dome-shaped curve is smaller than the first divergence load, the mode of instability changes from divergence to flutter.

The mode of instability for other combinations of support location, spring stiffness, and boundary conditions can be examined easily using the present formulation. Other complicating effects such as the presence of point mass, non-uniform mass and stiffness distributions can also be easily included in the present energy formulation.

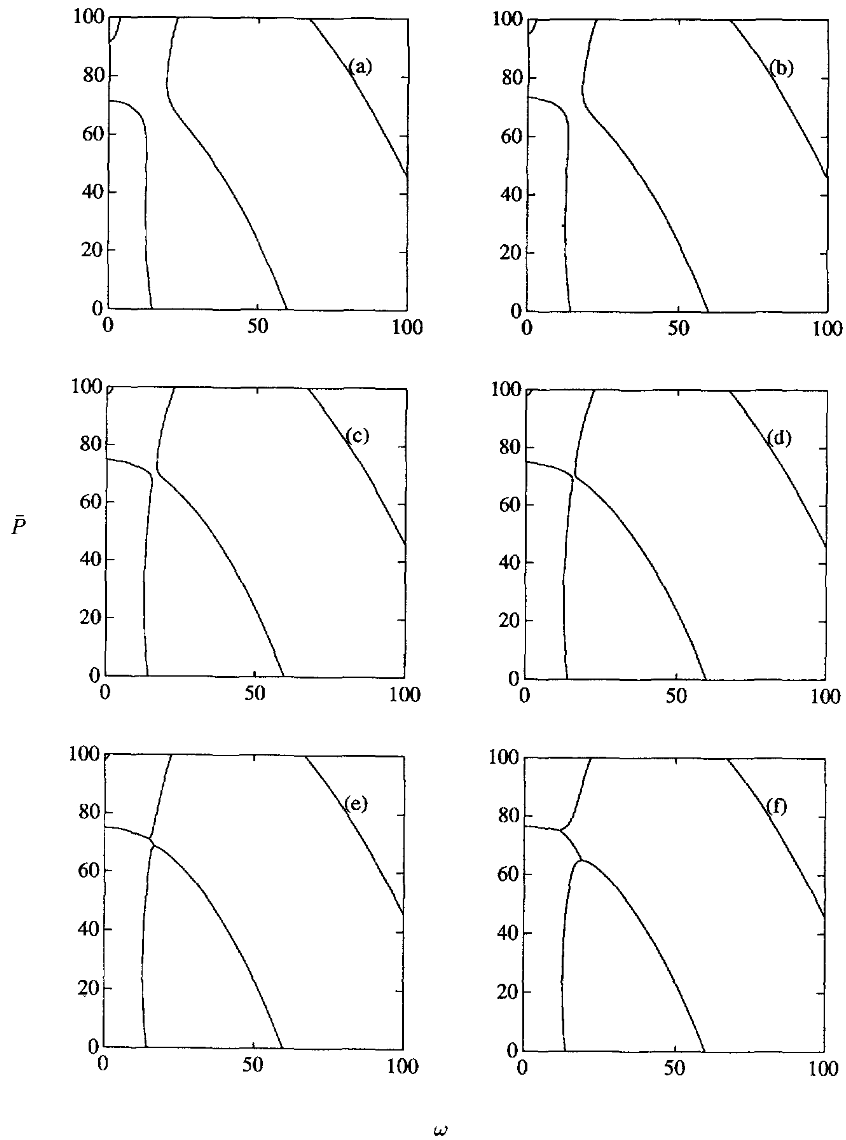

Fig. 7. The load–frequency curves for a cantilever rod with an intermediate spring support at $\bar{s} = 0.4$. (a) $\bar{k} = 200$; (b) $\bar{k} = 210$; (c) $\bar{k} = 218$; (d) $\bar{k} = 219$; (e) $\bar{k} = 220$; (f) $\bar{k} = 230$.

CONCLUSION

The present paper was intended to investigate the question of whether there is a jump in the follower force for the transition from flutter to divergence when the intermediate support of a cantilever beam subject to a follower force is shifted slightly. A different approach using the Lagrangian formulation and the assumed mode method is used to derive the matrix equation of motion. Jump phenomenon for the follower load is found to occur for both variations in the support location as well as the stiffness of the spring support. Detailed history of the changes in the load–frequency diagram due to variation in the support location and the stiffness of the spring support is presented to explain the occurrence of the jump phenomenon.

Damping has been shown to affect significantly the dynamic behavior of a cantilever rod subject to follower forces (see for example, Leipholz, 1980; Bolotin and Zhinzher, 1969). Kelvin–Voigt damping could be included in the present formulation as an additional term involving $\bar{\eta}\bar{\mathbf{K}}\dot{\mathbf{q}}$ in the equation of motion with η as the damping coefficient. Preliminary examination shows that for some cases, the modes of instability in the form of flutter or divergence without damping, are found to be unaffected by the presence of slight damping although there may be a sharp decrease in the first critical load for instability by flutter. However, for specific locations of the intermediate spring support, the presence of small amounts of damping may change the mode of instability from divergence to flutter with a sharp decrease in the critical follower load. Details of the study will be reported in due course. Damping is not included in the present analysis so as to simplify the stability analysis and to retain the elegance of the load–frequency diagrams.

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